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Kepler's Problem in Rotating Reference Frames Part 1: Prime Integrals, Vectorial Regularization

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This paper studies the Keplerian motion in rotating reference frames based on tensorial orthogonal and skew-symmetric maps. By using a time-regularization vectorial method, an exact solution to Keplerian noninertial motion is offered. A qualitative and quantitative comprehensive study is made. The paper generalizes the approaches to the inertial Keplerian motion presented by Levi-Civita and Kustaanheimo–Stiefel. A Sundman-like vectorial regularization is introduced to eliminate the singularity occurring in the attraction center. It transforms the strong nonlinear initial value problem with variable coefficients that describes the Keplerian motion in a noninertial reference frame into a linear one, with constant coefficients. A direct application to this approach is satellite relative dynamics.

Nomenclature

A^T	=	transpose of tensor (matrix) A
a	=	semimajor axis
$\mathbf{a}$	=	vectorial semimajor axis
$(\mathbf{a}, \mathbf{b}, \mathbf{c})$	=	triple product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$
b	=	semiminor axis
$\mathbf{b}$	=	vectorial semiminor axis
e	=	eccentricity
$\mathbf{e}$	=	vectorial eccentricity
$\mathbf{e_u}$	=	unit vector associated with vector $\mathbf{u}$: $\mathbf{e_u} = \frac{1}{u}\mathbf{u}$
$\mathbf{h}$	=	specific angular momentum
n	=	mean motion
p	=	semilatus rectum (conic parameter)
$\mathbf{R}_\omega$	=	rotation tensor with angular velocity $\boldsymbol{\omega}$
Re	=	set of real numbers
$\mathbf{r}$	=	position vector
t	=	time
u	=	magnitude of vector $\mathbf{u}$
$\mathbf{v}$	=	velocity vector
μ	=	gravitational parameter
ξ	=	specific energy
$\boldsymbol{\omega}$	=	angular velocity of the rotating reference frame
$\tilde{\boldsymbol{\omega}}$	=	skew-symmetric tensor associated with vector $\boldsymbol{\omega}$
$\cdot$	=	dot product
$\times$	=	cross product
$\triangleq$	=	"defined to be"
$\otimes$	=	dyadic (tensorial) product of vectors $\mathbf{a}$ and $\mathbf{b}$: $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} \triangleq (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

I. Introduction

THIS paper presents the exact solution to Kepler's problem in rotating reference frames. It is described by the Cauchy problem:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}, \qquad \mathbf{r}(t_0) = \mathbf{r} \quad \dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \tag{1}$$

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where t_0 is the initial moment of time and the vectorial map $\boldsymbol{\omega}$ is considered to be differentiable. In Eq. (1), the differentiable vectorial map represents the angular velocity of the rotating reference frame and the gravitational parameter, $\mu > 0$.

To the best of our knowledge, a comprehensive study of this problem is not included in classical theoretical mechanics textbooks [1–7], nor in orbital mechanics ones [8–10]. This is surprising, because this problem models numerous types of body motion such as satellite motion with respect to a frame positioned on Earth, the four-body problem, relative orbiting motion, and formation flying satellites [11–17].

In studying Kepler's problem in rotating reference frames different methods have been employed, either starting from the linear approximation of Eq. (1) near a known solution [18], or with the help of the Hamiltonian formalism [19]. By using a vectorial time regularization, the present paper provides a vectorial exact closed form solution to this problem. This regularizing procedure extends the approaches of Levi-Civita [20] and Kustaanheimo–Stiefel [21,22] in the case of rotating reference frames.

The paper is organized as follows. Section II introduces the main instrument for studying and solving Kepler's noninertial problem. Our methodology is based on the properties of real valued tensorial orthogonal and skew-symmetric maps [23,24]. The procedure is purely symbolic and it offers closed form solutions to Eq. (1) with time as an independent variable. The solutions do not depend on a particular coordinate system (Cartesian, cylindrical, spherical) chosen in the rotating reference frame.

The main result that gives the representation of the solution to Kepler's problem in rotating reference frames is presented in Sec. III. It links the inertial and noninertial problems together by means of an orthogonal tensorial map. It also opens the way for studying the following two main aspects: the shape of the trajectories, presented in Sec. IV.A, and the motion on the trajectories, presented in Sec. IV.B. The prime integrals of Kepler's problem in rotating reference frames allow the classification of the trajectories. They are analogous to the prime integrals in the inertial case, namely, the conservation of angular momentum, the conservation of energy, and the Laplace–Runge–Lenz vector.

A vectorial regularization introduced in Sec. IV.B.1 fundamentally helps to investigate the motion on the trajectories. An essential role in this study is played by a clone of the Laplace–Runge–Lenz vector.

The comprehensive analysis of Keplerian noninertial motion with respect to a rotating reference frame with arbitrary velocity $\boldsymbol{\omega}$ is revealed in Sec. IV.B.2. Explicit formulas are given in all seven possible cases, including the case when the initial position vector is collinear with the inertial velocity. A complete model for the Keplerian motion with respect to a rotating reference frame is given. Direct applications to our results involve the study of relative orbital dynamics.

II. Tensorial Considerations

The most important instrument in solving Kepler's problem in rotating reference frames and in studying the motion along the trajectories is introduced, namely, the orthogonal and skew-symmetric tensorial maps.

Let us denote by $\mathbf{V}_3$ the three-dimensional vectorial space and by $\mathbf{V}_3^{\mathrm{Re}}$ the set of the maps defined on the positive real semiaxis with values in $\mathbf{V}_3$. Let $\mathbf{SO}_3$ stand for the special orthogonal group of second-order tensors [5] (i.e., any tensor in $\mathbf{SO}_3$ satisfies $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$ and $\det(\mathbf{R}) = 1$, where $\mathbf{I}_3$ is the unit tensor). $\mathbf{SO}_3^{\mathrm{Re}}$ represents the set of maps defined on the positive real semiaxis with values in $\mathbf{SO}_3$. Similarly, denoting by $\mathbf{so}_3$ the Lie algebra of skew-symmetric second-order tensors [5] (i.e., $\tilde{\boldsymbol{\omega}} \in \mathbf{so}_3 \Leftrightarrow \tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^T = \mathbf{0}_3$), let $\mathbf{so}_3^{\mathrm{Re}}$ symbolize the set of maps defined on the positive real semiaxis with values in $\mathbf{so}_3$.

A. Orthogonal Tensorial Maps

This section presents the main notations used in the current paper as well as a brief study on tensorial orthogonal maps. This approach was first used in 1995 [24] and it relates the precession motion to proper orthogonal tensorial maps.

Let $\{F\}$ be a reference frame rotating with respect to the inertial frame $\{F^0\}$. A vector $\mathbf{r}^0$ related to frame $\{F^0\}$ is expressed in frame $\{F\}$ as

$$\mathbf{r} = \mathbf{R}\mathbf{r}^0 \tag{2}$$

where $\mathbf{R} \in \mathbf{SO}_3^{\mathrm{Re}}$ is a proper orthogonal tensorial map. Then

$$\dot{\mathbf{r}} = \dot{\mathbf{R}}\mathbf{r}^0 = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{r} \tag{3}$$

The tensorial map

$$\dot{\mathbf{R}}\mathbf{R}^T \triangleq \tilde{\boldsymbol{\omega}} \tag{4}$$

is skew symmetric, an element of $\mathbf{so}_3^{\mathrm{Re}}$. For an arbitrary tensorial map $\tilde{\boldsymbol{\omega}} \in \mathbf{so}_3^{\mathrm{Re}}$ the vectorial map $\boldsymbol{\omega} \in \mathbf{V}_3^{\mathrm{Re}}$ is associated with tensor $\tilde{\boldsymbol{\omega}}$ by the relation:

$$\tilde{\boldsymbol{\omega}}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}, \qquad (\forall)\ \mathbf{x} \in \mathbf{V}_3 \tag{5}$$

Vector $\boldsymbol{\omega}$ represents the instantaneous angular velocity of the rotating frame $\{F\}$ with respect to frame $\{F^0\}$.

The link between $\boldsymbol{\omega}$ and $\tilde{\boldsymbol{\omega}}$ may also be introduced by the matrix correspondence:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T \Leftrightarrow \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

where $\omega_i = \omega_i(t)$ are real valued maps and represent the coordinates of the vectorial map $\boldsymbol{\omega}$ in a positive oriented orthonormated basis.

We prove now that an arbitrary continuous vectorial map $\boldsymbol{\omega}$ (representing the instantaneous angular velocity) uniquely determines a proper orthogonal tensorial map. As it follows from Eq. (4), it is equivalent with proving that the equation

$$\dot{\mathbf{R}} = \tilde{\boldsymbol{\omega}}\mathbf{R}; \qquad \mathbf{R}(t_0) = \mathbf{I}_3 \tag{6}$$

has a unique solution $\mathbf{R}$ in $\mathbf{SO}_3^{\mathrm{Re}}$, $(\forall)\ \tilde{\boldsymbol{\omega}} \in \mathbf{so}_3^{\mathrm{Re}}$, $\tilde{\boldsymbol{\omega}}$ continuous on $[t_0, +\infty)$.

Indeed, by computing

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}^T\mathbf{R}) = \dot{\mathbf{R}}^T\mathbf{R} + \mathbf{R}^T\dot{\mathbf{R}} = -\mathbf{R}^T\tilde{\boldsymbol{\omega}}\mathbf{R} + \mathbf{R}^T\tilde{\boldsymbol{\omega}}\mathbf{R} = \mathbf{0}_3$$

it follows that $\mathbf{R}^T\mathbf{R}$ is constant, $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$. Because $\det\mathbf{R}$ is also a continuous function which satisfies $\det\mathbf{R} \in \{-1, 1\}$ and $(\det\mathbf{R})|_{t=t_0} = \det\mathbf{I}_3 = 1$, it follows that $\det\mathbf{R} = 1$. Since $\tilde{\boldsymbol{\omega}}$ is a continuous tensorial map, the solution to Eq. (6) is unique, $\mathbf{R} \in \mathbf{SO}_3^{\mathrm{Re}}$.

The unique solution to Eq. (6) is denoted $\mathbf{R}_{\boldsymbol{\omega}}$. The tensorial proper orthogonal map that models the rotation with angular velocity $-\boldsymbol{\omega}$ that will be denoted $\mathbf{R}_{-\boldsymbol{\omega}}$ is fundamental to the approach in this paper. From Eq. (6) it results that $\mathbf{R}_{-\boldsymbol{\omega}}$ is the solution to the differential equation:

$$\dot{\mathbf{Q}} + \tilde{\boldsymbol{\omega}}\mathbf{Q} = \mathbf{0}_3, \qquad \mathbf{Q}(t_0) = \mathbf{I}_3 \tag{7}$$

Remark 1: Equation (6) is the tensorial form of the famous Darboux equation [25]: finding the rotation tensor when knowing the instantaneous angular velocity. This problem is fundamental in attitude kinematics [18,26]. The link between the rotation tensorial map (also called orthogonal tensorial map) and the skew-symmetric tensor associated with the angular velocity vector is given by Eq. (6).

The solution to Eq. (7) helps to study the motion in rotating reference frames. Below we present some useful properties of the tensorial map $\mathbf{R}_{-\boldsymbol{\omega}}$.

1) $\mathbf{R}_{-\boldsymbol{\omega}}$ is invertible and $\mathbf{R}_{-\boldsymbol{\omega}}^{-1} = \mathbf{R}_{-\boldsymbol{\omega}}^T$;
2) $\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u} \cdot \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, $(\forall)\ \mathbf{u}, \mathbf{v} \in \mathbf{V}_3^{\mathrm{Re}}$;
3) $|\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}| = |\mathbf{u}|$, $(\forall)\ \mathbf{u} \in \mathbf{V}_3^{\mathrm{Re}}$;
4) $\mathbf{R}_{-\boldsymbol{\omega}}(\mathbf{u} \times \mathbf{v}) = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u} \times \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{v}$, $(\forall)\ \mathbf{u}, \mathbf{v} \in \mathbf{V}_3^{\mathrm{Re}}$;
5) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}_{-\boldsymbol{\omega}}^T\mathbf{u}) = \mathbf{R}_{-\boldsymbol{\omega}}^T(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u})$, $(\forall)\ \mathbf{u} \in \mathbf{V}_3^{\mathrm{Re}}$, differentiable;
6) $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{R}_{-\boldsymbol{\omega}}^T\mathbf{u}) = \mathbf{R}_{-\boldsymbol{\omega}}^T(\ddot{\mathbf{u}} + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) + \dot{\boldsymbol{\omega}} \times \mathbf{u})$, $(\forall)\ \mathbf{u} \in \mathbf{V}_3^{\mathrm{Re}}$, 2 times differentiable.

Properties (1)–(4) are valid for any proper orthogonal tensorial map [5,6]. Relations (5), (6) may be proved through direct computation by taking Eq. (7) into account.

B. Comments and Remarks

1) If $\boldsymbol{\omega}$ has a fixed direction, $\boldsymbol{\omega} = \omega\mathbf{u}$, with $\mathbf{u}$ its constant unit vector and ω: Re $\to$ Re, since $\tilde{\boldsymbol{\omega}}(t_1)\tilde{\boldsymbol{\omega}}(t_2) = \tilde{\boldsymbol{\omega}}(t_2)\tilde{\boldsymbol{\omega}}(t_1)$, $(\forall)\ t_{1,2} \in \mathrm{Re}$, then $\mathbf{R}_{-\boldsymbol{\omega}(t)}$ may be written as an exponential map [27] and it has the explicit expression:

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \exp\left(-\int_{t_0}^{t}\tilde{\boldsymbol{\omega}}(\xi)\,\mathrm{d}\xi\right) = \mathbf{I}_3 - \frac{\sin\varphi(t)}{\omega}\tilde{\boldsymbol{\omega}} + \frac{1-\cos\varphi(t)}{\omega^2}\tilde{\boldsymbol{\omega}}^2 \tag{8}$$

where $\varphi(t) = \int_{t_0}^{t}\omega(\xi)\,\mathrm{d}\xi$.

In this particular case, the inverse of tensor $\mathbf{R}_{-\boldsymbol{\omega}}$ (modeling the rotation with angular velocity $-\boldsymbol{\omega}$) is the tensor that models the rotation with angular velocity $\boldsymbol{\omega}$, that is,

$$\mathbf{R}_{-\boldsymbol{\omega}}^T = \mathbf{R}_{\boldsymbol{\omega}} \tag{9}$$

2) If $\boldsymbol{\omega}$ is constant, then $\mathbf{R}_{-\boldsymbol{\omega}(t)}$ has the explicit form:

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \exp[-(t-t_0)\tilde{\boldsymbol{\omega}}] = \mathbf{I}_3 - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}} + \frac{1-\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2 \tag{10}$$

Another expression for tensor $\mathbf{R}_{-\boldsymbol{\omega}}$, employed in computations later in this paper, uses the dyadic product:

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega}}{\omega^2} - \frac{\sin[\omega(t-t_0)]}{\omega}\tilde{\boldsymbol{\omega}} - \frac{\cos[\omega(t-t_0)]}{\omega^2}\tilde{\boldsymbol{\omega}}^2 \tag{11}$$

3) If vector $\boldsymbol{\omega}$ has a regular precession:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \mathbf{R}_{-\boldsymbol{\omega}_1}\boldsymbol{\omega}_2 \tag{12}$$

where $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are constant vectors, then $\mathbf{R}_{-\boldsymbol{\omega}}$ may be computed by means of

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \mathbf{R}_{-\boldsymbol{\omega}_1(t)}\mathbf{R}_{-\boldsymbol{\omega}_2(t)} \tag{13}$$

where the expressions of $\mathbf{R}_{-\boldsymbol{\omega}_1(t)}$, $\mathbf{R}_{-\boldsymbol{\omega}_2(t)}$ are similar to Eq. (10).

4) Equations (8), (10), and (13) give the time-explicit solution to the Darboux equation (7), namely, Eq. (8) when the angular velocity vector has constant direction, Eq. (10) when the angular velocity is constant, and Eq. (13) when the angular velocity has a regular precession. In the case when a time-explicit solution to Eq. (7) is not available, interpolation and spline approximation formulas may be applied [28].

5) If some vectorial map $\mathbf{u}$ satisfies

$$\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u} = \mathbf{0}; \qquad \mathbf{u}(t_0) = \mathbf{u}_0 \tag{14}$$

then $\mathbf{u} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0$ where $\mathbf{R}_{-\boldsymbol{\omega}}$ is the solution to Eq. (7).

To prove this apply tensor $\mathbf{R}^T_{-\boldsymbol{\omega}}$ to Eq. (14), leading to

$$\mathbf{R}^T_{-\boldsymbol{\omega}}(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}) = \mathbf{0} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{u}\right) = \mathbf{0} \tag{15}$$

It follows that $\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{u}$ is a constant vector, $\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{u} = \mathbf{u}_0$, equivalent with $\mathbf{u} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0$.

III. Representation Theorem to Kepler's Problem in Rotating Reference Frames. Prime Integrals

This represents the core of our paper. First the exact solution to Kepler's problem in rotating reference frames is delivered. The importance of the procedure presented in Theorem 1 resides not only in solving the particular problem under study here, but also in allowing its generalization to solve many other problems related to rotating reference frames. The prime integrals (conservation laws) of the Keplerian noninertial motion presented in the second part of this section generalize the prime integrals corresponding to Keplerian inertial motion.

A. Main Theorem

The next theorem uses the instruments presented previously. It gives the exact solution to Kepler's problem in a rotating reference frame that rotates with angular velocity $\boldsymbol{\omega}$ with respect to the inertial reference frame. This representation theorem will allow a comprehensive study of the Keplerian noninertial motion.

Theorem 1: The solution to the Cauchy problem

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}, \qquad \mathbf{r}(t_0) = \mathbf{r}_0$$
$$\dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \tag{16}$$

is obtained by applying the operator $\mathbf{R}_{-\boldsymbol{\omega}}$ to the solution to the Cauchy problem

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}, \qquad \mathbf{r}(t_0) = \mathbf{r}_0, \qquad \dot{\mathbf{r}}(t_0) = \mathbf{v}_0 + \omega_0 \times \mathbf{r}_0 \tag{17}$$

where $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(t_0)$.

Proof: By applying tensor $\mathbf{R}^T_{-\boldsymbol{\omega}}$ to Eq. (16) and by using the properties of this tensor, Eq. (16) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}\right] + \frac{\mu}{|\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}|^3}\left[\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}\right] = \mathbf{0}, \qquad \left[\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}\right]_{t=t_0} = \mathbf{r}_0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}\right]_{t=t_0} = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0 \tag{18}$$

Obviously, Eq. (18) is Eq. (17) written for the variable $\mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}$. It follows that if $\mathbf{r}$ is the solution to Eq. (17), then $\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}$ is the solution to Eq. (16). □

Remark 2: Theorem 1 shows that the Keplerian motion in a rotating reference frame can be decomposed into the following:

1) a classic Keplerian motion described by Eq. (17);

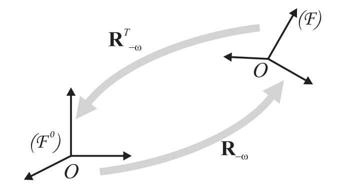

Fig. 1 The transformations between an inertial and a rotating reference frame.

2) a rotation with angular velocity $-\boldsymbol{\omega}$ of the reference frame associated with the previous motion.

As far as we know, no such representation theorem was given until now. It suggests a simple and effective way of studying the Keplerian motion in rotating reference frames. First we study the classic Keplerian motion described by Eq. (17), then tensor $\mathbf{R}_{-\boldsymbol{\omega}}$ is applied to the solution to this equation. Theorem 1 was formulated for the first time in another context [24].

Remark 3: The procedure suggested by Theorem 1 is purely symbolic and it can be easily generalized for solving most of the problems related to rotating reference frames. By applying the tensorial map $\mathbf{R}^T_{-\boldsymbol{\omega}}$ to Eq. (16), the problem is transferred in an inertial reference frame $\{F^0\}$. The problem is solved in this reference frame and its solution is transferred back in the rotating frame $\{F\}$ with the inverse tensorial operator that is $\mathbf{R}_{-\boldsymbol{\omega}}$. The inertial frame that appears in this approach (its existence is postulated by the inertia principle) plays only a catalyst role in this symbolic procedure. Finally, the solution to the noninertial problem is expressed in the original rotating reference frame (see also Fig. 1).

From a mathematical point of view, by using the change of variable

$$\mathbf{r} \rightarrow \mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r} \tag{19}$$

all noninertial terms present in the differential equation that describes the motion vanish. The transformation, similar to Laplace or Fourier transforms in the case of the linear scalar differential equations, modifies both the differential equation and the initial conditions. The initial condition $\dot{\mathbf{r}}(t_0) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0$ represents the inertial absolute velocity at $t = t_0$.

B. Prime Integrals of Kepler's Problem in Rotating Reference Frames

The prime integrals of the classic (inertial) Kepler's problem described by Eq. (17) are [1]:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) \triangleq \mathbf{h}_0 \tag{20}$$

(specific angular momentum conservation);

$$\frac{1}{2}\dot{\mathbf{r}}^2 - \frac{\mu}{r} = \frac{1}{2}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} \triangleq \xi \tag{21}$$

(specific energy conservation);

$$\frac{\dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}})}{\mu} - \frac{\mathbf{r}}{r} = \frac{(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) \times [\mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)]}{\mu} - \frac{\mathbf{r}_0}{r_0}$$
$$\triangleq \mathbf{e}_0 \tag{22}$$

(the Laplace–Runge–Lenz vector).

From Theorem 1 and by using the properties of the tensorial map $\mathbf{R}_{-\boldsymbol{\omega}}$, it results that the prime integrals of Kepler's problem in a rotating reference frame may be obtained as follows: in Eqs. (20–22) replace $\mathbf{r} \rightarrow \mathbf{R}^T_{-\boldsymbol{\omega}}\mathbf{r}$ and therefore $\dot{\mathbf{r}} \rightarrow \mathbf{R}^T_{-\boldsymbol{\omega}}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})$. By taking into account the properties (1)–(5) of $\mathbf{R}^T_{-\boldsymbol{\omega}}$ in Sec. II.A, it follows that the prime integrals of Kepler's problem in a rotating reference frame are

$$\mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{h}_0 \triangleq \mathbf{h} \tag{23}$$

(analogous to angular momentum conservation);

$$\frac{1}{2}\dot{\mathbf{r}}^2 + (\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \frac{\mu}{r} = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} \triangleq \xi \tag{24}$$

(analogous to energy conservation);

$$\frac{(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \times [\mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})]}{\mu} - \frac{\mathbf{r}}{r} = \mathbf{R}_{-\omega}\mathbf{e}_0 \triangleq \mathbf{e} \tag{25}$$

(analogous to Laplace–Runge–Lenz vector), where

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}(t_0) \tag{26}$$

$$\mathbf{h}_0 = \mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) \tag{27}$$

$$\mathbf{e}_0 = \frac{(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) \times [\mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)]}{\mu} - \frac{\mathbf{r}_0}{r_0} \tag{28}$$

Remark 4: Relations (23–25) are analogous to Eqs. (20–22), which can be easily found by making $\boldsymbol{\omega} = \mathbf{0}$.

IV. Keplerian Motion in Rotating Reference Frames

A. Trajectories

Let $\mathbf{h}_0$, ξ, and $\mathbf{e}_0$ be the prime integrals of Eq. (17), the classic Kepler problem with respect to an inertial frame. The corresponding prime integrals of the equation modeling the Keplerian motion with respect to a rotating reference frame are essential in studying the trajectories. Vectors $\mathbf{h} = \mathbf{R}_{-\omega}\mathbf{h}_0$ and $\mathbf{e} = \mathbf{R}_{-\omega}\mathbf{e}_0$ become rotating vectors with angular velocity $-\boldsymbol{\omega}$. Together with the prime integral ξ, they give the nature of the trajectories, that may be plane or spatial curves, bounded or unbounded, depending on the initial conditions and on vector $\boldsymbol{\omega}$.

The particular situations that may arise are amply discussed further.

1) In relation (23) two cases may occur:

a) When $\mathbf{h}_0 \neq \mathbf{0}$, it results as $\mathbf{h} \neq \mathbf{0}$ and $\mathbf{r} \cdot \mathbf{h} = \mathbf{0}$. The motion can be decomposed into the following:

i) a classic planar Keplerian motion described by Eq. (17);

ii) a rotation with angular velocity $-\boldsymbol{\omega}$ of the reference frame associated with the classic Keplerian motion.

The plane of the classic Keplerian motion at the initial moment $t = t_0$ is denoted $\boldsymbol{\Pi}(t_0)$. It passes through the attraction center and has the normal vector $\mathbf{h}_0$. It starts rotating with angular velocity $-\boldsymbol{\omega}$ around the attraction center. Vector $\mathbf{h}$ is normal to this plane. At the moment of time t, the vectorial equation of plane $\boldsymbol{\Pi}(t)$ is given by

$$\mathbf{r} \cdot \mathbf{h} = \mathbf{0}$$

b) When $\mathbf{h}_0 = \mathbf{0}$, it follows that $\mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) = \mathbf{0}$. The motion can be decomposed into the following:

i) a rectilinear Keplerian motion;

ii) a rotation with angular velocity $-\boldsymbol{\omega}$ of the reference frame associated with the classic rectilinear Keplerian motion.

Let $\mathbf{u}$ denote the unit vector of the rotating straight line where the motion occurs. There exists a differentiable function x: Re → Re such that $\mathbf{r}(t) = x(t)\mathbf{u}(t)$. By using Theorem 1, the rectilinear motion is described by

$$\ddot{x} + \frac{\mu}{|x|^3}x = 0, \qquad x(t_0) = r_0, \qquad \dot{x}(t_0) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0} \tag{29}$$

2) Relation (24) represents the generalized energy conservation. Denoting by

$$E_{\text{kin}} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 \tag{30}$$

the *kinetic energy* and by

$$V = m\left[(\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \frac{\mu}{r}\right] \tag{31}$$

the *generalized potential energy*, the energy conservation law is obtained as

$$\xi = \frac{E_{\text{kin}} + V}{m} = \text{const} \tag{32}$$

In general, $V = V(t, \mathbf{r}, \dot{\mathbf{r}})$. Two particular cases may be emphasized.

a) In the case when $\boldsymbol{\omega}$ has constant direction, $\mathbf{R}_{-\omega}\boldsymbol{\omega} = \boldsymbol{\omega}$. By dot multiplying relation (23) with $\boldsymbol{\omega}$, it follows that

$$(\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) = \omega \cdot \mathbf{R}_{-\omega}\mathbf{h}_0 = \boldsymbol{\omega} \cdot \mathbf{h}_0$$

Therefore

$$V = m\left[\boldsymbol{\omega} \cdot \mathbf{h}_0 - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \frac{\mu}{r}\right] \tag{33}$$

so that $V = V(t, \mathbf{r})$.

b) When $\boldsymbol{\omega}$ is constant, $\boldsymbol{\omega} = \boldsymbol{\omega}_0$,

$$V = m\left[\boldsymbol{\omega}_0 \cdot \mathbf{h}_0 - \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r})^2 - \frac{\mu}{r}\right] \tag{34}$$

In this case $V = V(\mathbf{r})$ (i.e., it represents the classic potential energy).

3) Starting from relation (25), many interesting properties of the motion can be found. First, let us remark that from $\mathbf{e} = \mathbf{R}_{-\omega}\mathbf{e}_0$ it follows that vector $\mathbf{e}$ has a constant magnitude. Through direct computations, the magnitude of vector $\mathbf{e}$ is obtained as

$$e = \sqrt{1 + \frac{2h^2\xi}{\mu^2}} \tag{35}$$

where $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0}$ and $h^2 = h_0^2 = [\mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)]^2$.

Two major cases may occur.

a) The case when $\mathbf{h}_0 \neq \mathbf{0}$.

By dot multiplying Eq. (25) with $\mathbf{h}$, we get $\mathbf{h} \cdot \mathbf{e} = \mathbf{0}$, showing that vector $\mathbf{e}$ is also situated in plane $\boldsymbol{\Pi}(t)$.

By dot multiplying Eq. (25) with $\mathbf{r}$, we get

$$\mathbf{r} \cdot \mathbf{e} + r = \frac{1}{\mu}[\mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})]^2 = \frac{1}{\mu}\mathbf{h}_0^2 \triangleq p \Rightarrow r = \frac{p}{1 + \mathbf{e} \cdot \mathbf{e}_{\mathbf{r}}} \tag{36}$$

In a reference frame where vector $\mathbf{e}$ is immobile, namely, in plane $\boldsymbol{\Pi}(t)$, Eq. (36) describes a conic with the *semilatus rectum p* and eccentricity e. In this reference frame, $\mathbf{e}$ represents the Laplace–Runge–Lenz vector (see Fig. 2).

The bounded or unbounded character of the trajectory is discussed further.

i) If $e < 1$ [equivalent to $\xi < 0$ in relation (35)], the trajectory in plane $\boldsymbol{\Pi}(t)$ is an ellipse, therefore it is bounded. As plane $\boldsymbol{\Pi}(t)$ has a fixed point, that is the attraction center, the spatial region where the trajectory is located represents the region between two concentric spheres, one with radius $p/(1 + e)$ and one with radius $p/(1 - e)$ (see Fig. 3).

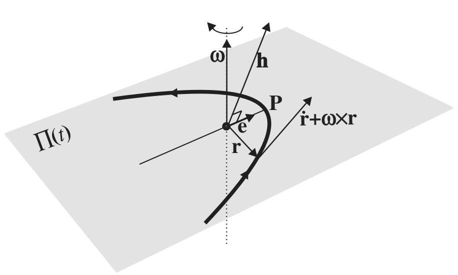

Fig. 2 The generic Keplerian motion in a rotating reference frame for $\mathbf{h}_0 \neq \mathbf{0}$.

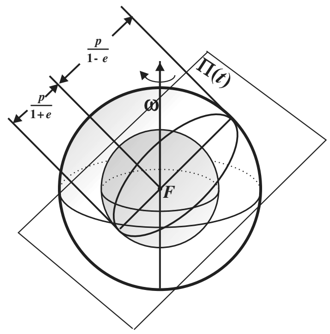

Fig. 3 For $\mathbf{h}_0 \neq 0$, $e < 1$, the trajectory is bounded between two concentric spheres.

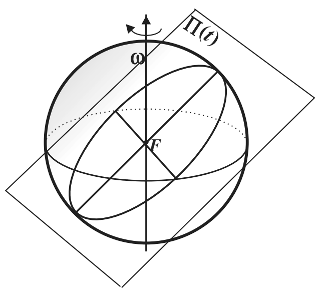

Fig. 4 For $\mathbf{h}_0 \neq 0$, $e = 0$, the trajectory is a spherical curve.

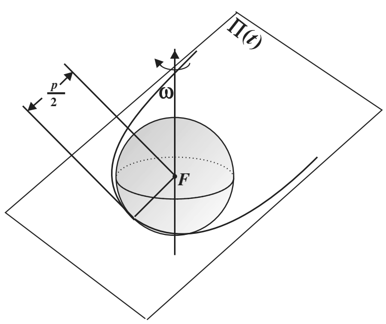

Fig. 5 For $\mathbf{h}_0 \neq 0$, $e = 1$, the trajectory is unbounded outside a sphere.

In the particular case when $e = 0$, the trajectory in plane $\mathbf{\Pi}(t)$ is a circle with its center in the attraction point and the radius p. Then the body trajectory is a spherical curve, generated by the rigid rotation of this circle with angular velocity $-\boldsymbol{\omega}$ (see Fig. 4).

ii) If $e = 1$ [$\xi = 0$, see (35)], the trajectory is unbounded, because it is a parabola in plane $\mathbf{\Pi}(t)$; the trajectory is located outside a sphere that is centered in the attraction point and has the radius $\frac{p}{2}$ (see Fig. 5).

iii) If $e > 1$ [$\xi > 0$, see (35)], the trajectory is unbounded, because it is a hyperbola in plane $\mathbf{\Pi}(t)$; the trajectory is situated outside a sphere centered in the attraction point and with radius $p/(1+e)$ (see Fig. 6).

Besides the study of the boundedness of the trajectory that was presented above, in the particular case when $\mathbf{h}_0 \neq \mathbf{0}$ another prime

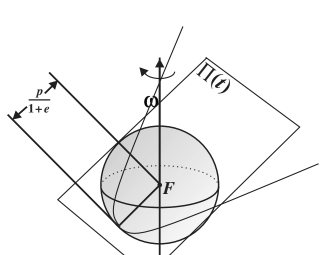

Fig. 6 For $\mathbf{h}_0 \neq 0$, $e > 1$, the trajectory is unbounded outside a sphere.

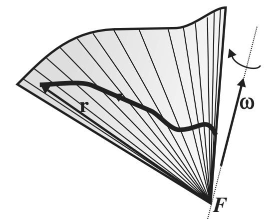

Fig. 7 For $\mathbf{h}_0 = 0$, the trajectory is situated on a conical surface.

integral of the motion may be revealed further. By cross multiplying Eq. (25) with $\mathbf{h}$ and denoting $\mathbf{e_r} \triangleq \frac{1}{r}\mathbf{r}$, $\alpha \triangleq \frac{\mu}{h^2}\mathbf{h}$, we get $\alpha \cdot \mathbf{e} = \alpha_0 \cdot \mathbf{e}_0 = 0$ leading to

$$\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r} = \alpha \times (\mathbf{e} + \mathbf{e}_r) \tag{37}$$

Equation (37) shows that the hodograph of the vectorial map $\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}$ is a curve that can be decomposed into the following:

– a circular section in plane $\mathbf{\Pi}(t)$, because $\boldsymbol{\alpha}$ is normal to this plane and vectors $\mathbf{e}$ and $\mathbf{e}_r$ are included in $\mathbf{\Pi}(t)$;

– the rotation of plane $\mathbf{\Pi}(t)$ with angular velocity $-\boldsymbol{\omega}$.

b) The case when $\mathbf{h}_0 = \mathbf{0}$.

From Eq. (25) it follows that $\mathbf{e} = -\frac{\mathbf{r}}{r}$, therefore vector $\mathbf{e}$ has constant unitary magnitude and the opposite sense to the position vector $\mathbf{r}$. Equation (25) leads to $\mathbf{r} = -r\mathbf{R}_{-\omega}\mathbf{e}_0$, showing that the trajectory is a curve situated on the conical surface generated by the rotation with angular velocity $-\boldsymbol{\omega}$ of vector $\mathbf{e}_0$ (see Fig. 7).

Remark 5: If vector $\boldsymbol{\omega}$ has a fixed direction, $\boldsymbol{\omega} = \omega\mathbf{u}$, with $\mathbf{u}$ the constant unit vector and ω: Re $\to$ Re, then plane $\mathbf{\Pi}(t)$ has an irregular precession motion with angular velocity $-\boldsymbol{\omega}$. Vectors $\mathbf{e}$ and $\mathbf{h}$ have the following explicit forms:

$$\mathbf{h} = \frac{\mathbf{h}_0 \cdot \boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin\varphi(t)}{\omega}\boldsymbol{\omega} \times \mathbf{h}_0 - \frac{\cos\varphi(t)}{\omega^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{h}_0) \tag{38}$$

$$\mathbf{e} = \frac{\mathbf{e}_0 \cdot \boldsymbol{\omega}}{\omega^2}\boldsymbol{\omega} - \frac{\sin\varphi(t)}{\omega}\boldsymbol{\omega} \times \mathbf{e}_0 - \frac{\cos\varphi(t)}{\omega^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}_0) \tag{39}$$

where $\varphi(t) = \int_{t_0}^{t} \omega(\xi)\,\mathrm{d}\xi$. Vectors $\mathbf{e}$ and $\mathbf{h}$ also have an irregular precession motion with angular velocity $-\boldsymbol{\omega}$. They both "sweep" the surface of a right circular cone. This remark is relevant to the shape of the trajectory (see Fig. 8).

Remark 6: If vector $\boldsymbol{\omega}$ is constant, $\boldsymbol{\omega} = \omega\mathbf{u}$, with $\mathbf{u}$ a constant unit vector and $\omega \neq 0$ a constant real number; then plane $\mathbf{\Pi}(t)$ has a regular precession motion with angular velocity $-\boldsymbol{\omega}$. Vectors $\mathbf{e}$ and $\mathbf{h}$ have the following explicit form:

$$\mathbf{h} = \frac{\mathbf{h}_0 \cdot \omega}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t - t_0)]}{\omega}\boldsymbol{\omega} \times \mathbf{h}_0 - \frac{\cos[\omega(t - t_0)]}{\omega^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{h}_0) \tag{40}$$

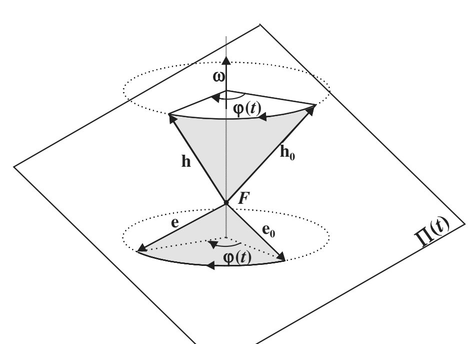

Fig. 8 When ω has a fixed direction, vectors h and e have an irregular precession motion.

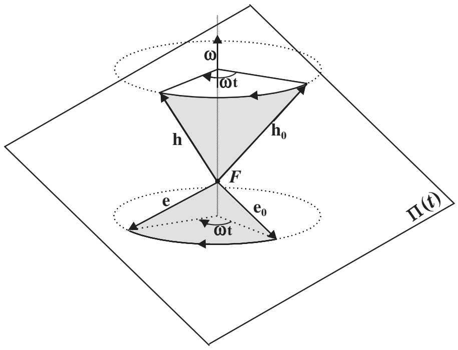

Fig. 9 When ω is constant, vectors h and e have a regular precession motion.

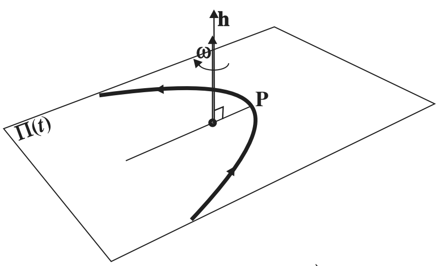

Fig. 10 The trajectory when $\mathbf{h} = \overrightarrow{\text{const}}$ is a plane curve.

$$\mathbf{e} = \frac{\mathbf{e}_0 \cdot \omega}{\omega^2}\boldsymbol{\omega} - \frac{\sin[\omega(t - t_0)]}{\omega}\boldsymbol{\omega} \times \mathbf{e}_0 - \frac{\cos[\omega(t - t_0)]}{\omega^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}_0) \tag{41}$$

Vectors **e** and **h** also have a regular precession motion with angular velocity $-\boldsymbol{\omega}$, sweeping the surface of a right circular cone (see Fig. 9).

Remark 7: If vector **h** is constant and nonzero, the trajectory is a plane curve. In this case vector $\boldsymbol{\omega}$ has a fixed direction. Plane $\boldsymbol{\Pi}(t)$ has a fixed point, the attraction center, and it rotates with angular velocity $-\boldsymbol{\omega}$ around a fixed axis (see Fig. 10).

B. Motion on the Trajectories

1. Vectorial Regularization of Kepler's Problem in Rotating Reference Frames

The regularization of Kepler's problem has a long history. From the regularization methods that were introduced, we only mention those of Levi-Civita in 1920 using complex numbers, for the planar case [20], and of Kustaanheimo–Stiefel in 1964 for the spatial case [21,22], based on spinors. The vectorial time regularization introduced in our paper unifies the approaches to Kepler's problem for both the inertial and noninertial case.

In Eq. (1) we introduce a new variable $\tau = \tau(t)$ such as for any differentiable map $\mathbf{u} \in V_3^{\mathrm{Re}}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{u} = r(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}) \tag{42}$$

where r is the magnitude of the solution to Eq. (1). Then $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r} = r(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})$. By dot multiplying this relation with **r**, we get $\mathbf{r} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r} = r(\mathbf{r} \cdot \dot{\mathbf{r}})$, and further $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r}^2 = r\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^2$. It follows that $\mathrm{d}t = r\,\mathrm{d}\tau$, which proves that the time transformation $t \to \tau = \tau(t)$ is a Sundman transformation [29].

From Sec. II.B, by using Remark 5, it follows that if

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{u} = \mathbf{0}$$

then $\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u} = \mathbf{0}$ and $\mathbf{u} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0$ where $\mathbf{u}_0 = \mathbf{u}(t_0)$.

Let us consider the following substitution:

$$t(\tau) = \int_0^\tau r(\xi)\,\mathrm{d}\xi + t_P \tag{43}$$

When $\mathbf{h}_0 \neq \mathbf{0}$, t_P denotes the time of periapsis passage in a reference frame where plane $\boldsymbol{\Pi}(t)$ described in Sec. III is immobile. When $\mathbf{h}_0 = \mathbf{0}$, t_P denotes the impact moment with the attraction center. The substitution made in (43) regularizes Eq. (1).

Noticing that

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{r} = \frac{\mathrm{d}}{\mathrm{d}\tau}[r(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})]$$

and making all computations [by taking Eqs. (24) and (25) into account]:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{r} = 2\xi\mathbf{r} - \mu\mathbf{e} \tag{44}$$

Equation (1) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{r} - 2\xi\mathbf{r} = -\mu\mathbf{e} \tag{45}$$

$$\mathbf{r}|_{\tau=0} = \begin{cases} \frac{h^2}{\mu(1+e)}\frac{\mathbf{e}}{e}, & \mathbf{e} \neq \mathbf{0} \\ \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_0, & \mathbf{e} = \mathbf{0} \end{cases}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r}\right|_{\tau=0} = \begin{cases} \frac{1}{e}\mathbf{h} \times \mathbf{e}, & \mathbf{e} \neq \mathbf{0} \\ r_0\mathbf{R}_{-\boldsymbol{\omega}}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0), & \mathbf{e} = \mathbf{0} \end{cases}$$

When $\mathbf{h}_0 = \mathbf{0}$, then $\mathbf{r}|_{\tau=0} = \mathbf{0}$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r}|_{\tau=0} = \mathbf{0}$. In this case, $\tau = 0$ marks the collision "moment" between the particle and the attraction center. Equation (45) is a vectorial regularization of (1), eliminating the singularity for $\mathbf{r} = \mathbf{0}$.

The vectorial regularization introduced above transforms the nonlinear differential equation with variable coefficients (1) into a linear one with constant coefficients, namely, (45).

2. Law of Motion and Velocity

This section comprises a complete study of the Keplerian noninertial motion on the trajectory. Explicit formulas are derived for the law of motion and for the velocity. By using the substitution introduced in Sec. IV.B.1, explicit laws of motion are next given for the Keplerian motion in rotating reference frames. By solving a second-order linear differential equation, three main cases occur, depending on the coefficients and the initial conditions. The nature of the trajectories, bounded or unbounded, is given by the sign of ξ that was introduced in (24). In all the cases that will be presented further on, the following algorithm is applied:

1) Eq. (45) is solved and an explicit solution is given (in the variable τ);

2) the magnitude of the radius vector is computed and replaced in Eq. (43);

3) the velocity is computed by using

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r} = r(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \Rightarrow \dot{\mathbf{r}} = \frac{1}{r}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{r}\right) - \boldsymbol{\omega} \times \mathbf{r} \tag{46}$$

4) the moment of time t_P is determined from the initial conditions;

5) an implicit relation between τ and t is given by using $t - t_P = \int_0^\tau r(\xi)\,\mathrm{d}\xi$. When $\xi \neq 0$, the map $E[\tau(t)] = \sqrt{2|\xi|}\tau(t)$ is implicitly defined. It is analogous to the *eccentric anomaly* from the classic Keplerian motion. Kepler's equations are defined in a unitary way in each case.

All expressions in the case when $\xi \neq 0$ are given depending on three new prime integrals that are based on **h**, ξ, and **e**:

$$\mathbf{a} = \begin{cases} \frac{\mu}{2e|\xi|}\mathbf{e}, & \mathbf{e} \neq \mathbf{0} \\ \mathbf{R}_{-\omega(t)}\mathbf{r}_0, & \mathbf{e} = \mathbf{0} \end{cases}$$
$$\mathbf{b} = \begin{cases} \frac{1}{e\sqrt{2|\xi|}}\mathbf{h} \times \mathbf{e}, & \mathbf{e} \neq \mathbf{0} \\ \frac{1}{n}\mathbf{R}_{-\omega(t)}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0), & \mathbf{e} = \mathbf{0} \end{cases}; \qquad n = \frac{(2|\xi|)^{\frac{3}{2}}}{\mu} \tag{47}$$

When $\xi \neq 0$, vectors $\mathbf{a}$ and $\mathbf{b}$ represent the vectorial semimajor and semiminor axis of the trajectory (an ellipse or a hyperbola) in plane $\boldsymbol{\Pi}(t)$. Here n is analogous to the *mean motion* from the inertial Keplerian situation.

As follows from Theorem 1, $\mathbf{a} = \mathbf{R}_{-\omega(t)}\mathbf{a}_0$ and $\mathbf{b} = \mathbf{R}_{-\omega(t)}\mathbf{b}_0$, where

$$\mathbf{a}_0 = \begin{cases} \frac{\mu}{2e|\xi|}\mathbf{e}_0, & \mathbf{e}_0 \neq \mathbf{0} \\ \mathbf{r}_0, & \mathbf{e}_0 = \mathbf{0} \end{cases}; \qquad \mathbf{b}_0 = \begin{cases} \frac{1}{e\sqrt{2|\xi|}}\mathbf{h}_0 \times \mathbf{e}_0, & \mathbf{e}_0 \neq \mathbf{0} \\ \frac{1}{n}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) & \mathbf{e}_0 = \mathbf{0} \end{cases} \tag{48}$$

Particular computations will be done in some cases, depending on particular conditions imposed. An interesting relation between the new prime integrals and the initial ones is

$$n(\mathbf{a} \times \mathbf{b}) = \mathbf{h} \tag{49}$$

a. Case of Negative Specific Energy.

a-i: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} < 0$ and $\mathbf{h}_0 \neq \mathbf{0}$, $e \in (0, 1)$ the motion on the trajectory is described by (see also Fig. 11)

$$\mathbf{r}(t) = [\cos E(t) - e]\mathbf{R}_{-\omega(t)}\mathbf{a}_0 + \sin E(t)\mathbf{R}_{-\omega(t)}\mathbf{b}_0$$
$$\mathbf{v}(t) = \frac{n}{1 - e\cos E(t)}[-\sin E(t)\mathbf{R}_{-\omega(t)}\mathbf{a}_0 + \cos E(t)\mathbf{R}_{-\omega(t)}\mathbf{b}_0]$$
$$- [\cos E(t) - e][\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{a}_0] - \sin E(t)[\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{b}_0] \tag{50}$$

where $t \in [t_0, +\infty)$ and $E(t)$ is defined from

$$E(t) - e\sin E(t) = n(t - t_0) + E_0 - e\sin E_0, \qquad t \in [t_0, +\infty) \tag{51}$$

with $E_0 \in [0, 2\pi)$ given by:

$$\cos E_0 = \frac{1}{e}\left(1 - n\frac{r_0}{\sqrt{2|\xi|}}\right)$$
$$\sin E_0 = n\frac{(\mathbf{v}_0 \cdot \mathbf{r}_0)}{2e|\xi|}\left(1 - \frac{\boldsymbol{\omega}_0 \cdot \mathbf{h}_0}{\mu}r_0\right) \tag{52}$$

where

$$e = \sqrt{1 - \frac{2h^2|\xi|}{\mu^2}}$$

and

$$n = \frac{(2|\xi|)^{\frac{3}{2}}}{\mu}$$

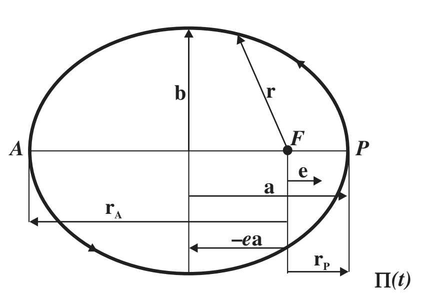

Fig. 11 The motion in plane $\boldsymbol{\Pi}(t)$ when $\xi < 0$, $\mathbf{h}_0 \neq \mathbf{0}$.

a-ii: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} < 0$, $\mathbf{h}_0 \neq \mathbf{0}$, $e = 0$ the motion on the trajectory is described by

$$\mathbf{r}(t) = \cos[n(t - t_0)]\mathbf{R}_{-\omega(t)}\mathbf{r}_0 + \frac{1}{n}\sin[n(t - t_0)]$$
$$\times \mathbf{R}_{-\omega(t)}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)$$
$$\mathbf{v}(t) = -n\sin[n(t - t_0)]\mathbf{R}_{-\omega(t)}\mathbf{r}_0 + \cos[n(t - t_0)]\mathbf{R}_{-\omega(t)}$$
$$\times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0) - \cos[n(t - t_0)][\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{r}_0]$$
$$- \frac{1}{n}\sin[n(t - t_0)][\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}(\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0)] \tag{53}$$

where $t \in [t_0, +\infty)$ and $n = (2|\xi|)^{3/2}/\mu$.

a-iii: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} < 0$ and $\mathbf{h}_0 = \mathbf{0}$, the motion on the trajectory is described by

$$\mathbf{r}(t) = \frac{\sqrt{2|\xi|}}{n}[\cos E(t) - 1]\mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0}$$
$$\mathbf{v}(t) = \frac{\sqrt{2|\xi|}\sin E(t)}{1 - \cos E(t)}\mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0} - \frac{\sqrt{2|\xi|}}{n}[\cos E(t) - 1]$$
$$\times \left[\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0}\right] \tag{54}$$

where $t \in [t_0, t_P + \frac{2\pi}{n})$, $n = (2|\xi|)^{3/2}/\mu$. $E(t)$ is defined by

$$E(t) - \sin E(t) = n(t - t_0) + E_0 - \sin E_0 \tag{55}$$

and $E_0 \in [0, 2\pi)$ is given by

$$\cos E_0 = 1 - n\frac{r_0}{\sqrt{2|\xi|}}; \qquad \sin E_0 = n\frac{(\mathbf{v}_0 \cdot \mathbf{r}_0)}{2e|\xi|} \tag{56}$$

The moment t_P is computed from

$$t_P = t_0 - \frac{1}{n}(E_0 - \sin E_0) \tag{57}$$

The periodicity of the motion for $\xi < 0$, $\mathbf{h}_0 \neq 0$: Because $E(t)$ is periodic, with the main period:

$$T = \frac{2\pi}{n} \tag{58}$$

we state that a necessary condition for the motion to be periodic is that function $\mathbf{R}_{-\omega}$ is periodic. If $\mathbf{R}_{-\omega}$ is periodic and its period is T_1 then the condition for motion periodicity is that number $\frac{T}{T_1}$ be a rational one, that is,

$$\frac{T}{T_1} = \frac{k}{q}, \qquad k, q \in \mathbb{N}^*, \qquad \text{relative prime} \tag{59}$$

It follows that the main period of the motion is $T_0 = qT = kT_1$. If $\boldsymbol{\omega} = \overrightarrow{\text{const}}$, the map $\mathbf{R}_{-\omega}$ is periodic with the main period $T_1 = \frac{2\pi}{\omega}$. Therefore the motion is periodic if and only if $\frac{\omega}{n}$ is a rational number. In this case, the trajectory is a closed spatial curve.

b. Case of Zero Specific Energy.

b-i: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} = 0$ and $\mathbf{h}_0 \neq \mathbf{0}$, then the motion on the trajectory is described by (see also Fig. 12)

$$\mathbf{r}(t) = \frac{1}{2}[p - \mu\tau^2(t)]\mathbf{R}_{-\omega(t)}\mathbf{e}_0 + \tau(t)\mathbf{R}_{-\omega(t)}(\mathbf{h}_0 \times \mathbf{e}_0)$$
$$\mathbf{v}(t) = \frac{2}{p + \mu\tau^2(t)}[-\mu\tau(t)\mathbf{R}_{-\omega(t)}\mathbf{e}_0 + \mathbf{R}_{-\omega(t)}(\mathbf{h}_0 \times \mathbf{e}_0)]$$
$$- \frac{1}{2}[p - \mu\tau^2(t)][\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{e}_0] - \tau(t)[\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}(\mathbf{h}_0 \times \mathbf{e}_0)] \tag{60}$$

with $t \in [t_0, +\infty)$ and $\tau(t)$ and t_P given by

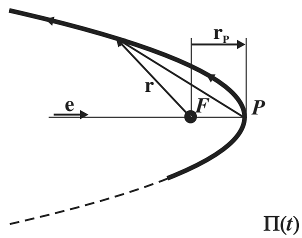

Fig. 12 The motion in plane $\Pi(t)$ when $\xi = 0$, $\mathbf{h}_0 \neq \mathbf{0}$.

$$\tau(t) = \frac{1}{\sqrt[3]{\mu}}\sqrt[3]{3(t_P - t) + \sqrt{9(t_P - t)^2 + p^3/\mu}} + \frac{1}{\sqrt[3]{\mu}}\sqrt[3]{3(t_P - t) - \sqrt{9(t_P - t)^2 + p^3/\mu}}$$
$$t_P = t_0 - \frac{1}{2}\left[p\tau(t_0) + \frac{\mu}{3}\tau^3(t_0)\right]$$
$$\tau(t_0) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\mu^2}[\mu - r_0(\boldsymbol{\omega}_0 \cdot \mathbf{h}_0)] \tag{61}$$

b-ii: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} = 0$ and $\mathbf{h}_0 = \mathbf{0}$, then the motion is described by

$$\mathbf{r}(t) = \frac{1}{2}\mu^{\frac{1}{3}}[6(t - t_P)]^{\frac{2}{3}}\mathbf{R}_{-\omega}\frac{\mathbf{r}_0}{r_0}$$
$$\mathbf{v}(t) = 2\left[\frac{6(t - t_P)}{\mu}\right]^{-\frac{1}{3}}\mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0} - \frac{1}{2}\mu^{\frac{1}{3}}[6(t - t_P)]^{\frac{2}{3}}\left[\boldsymbol{\omega} \times \mathbf{R}_{-\omega}\frac{\mathbf{r}_0}{r_0}\right]$$
$$t_P = t_0 - \frac{(\mathbf{r}_0 \cdot \mathbf{v}_0)^{\frac{3}{2}}}{6\mu^2} \tag{62}$$

The formulas (62) work for $t \in [t_0, t_P)$ if $\mathbf{r}_0 \cdot \mathbf{v}_0 < 0$ and for $t \in [t_0, +\infty)$ if $\mathbf{r}_0 \cdot \mathbf{v}_0 > 0$.

c. Case of Positive Specific Energy.

c-i: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} > 0$ and $\mathbf{h}_0 \neq \mathbf{0}$, the motion on the trajectory is described by (see also Fig. 13)

$$\mathbf{r}(t) = [e - \cosh E(t)]\mathbf{R}_{-\omega(t)}\mathbf{a}_0 + \sinh E(t)\mathbf{R}_{-\omega(t)}\mathbf{b}_0$$
$$\mathbf{v}(t) = \frac{n}{e\cosh E(t) - 1}[-\sinh E(t)\mathbf{R}_{-\omega(t)}\mathbf{a}_0 + \cosh E(t)\mathbf{R}_{-\omega(t)}\mathbf{b}_0] - [e - \cosh E(t)][\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{a}_0] - \sinh E(t)[\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\mathbf{b}_0] \tag{63}$$

with $t \in [t_0, +\infty)$ and $E(t)$ defined by the implicit functional equation:

$$e\sinh E(t) - E(t) = n(t - t_0) + e\sinh E_0 - E_0, \qquad t \in [t_0, +\infty)$$

where

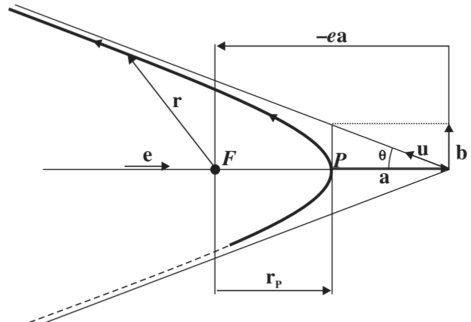

Fig. 13 The motion in plane $\Pi(t)$ when $\xi > 0$, $\mathbf{h}_0 \neq \mathbf{0}$.

$$E_0 = \sinh^{-1}\left\{n\frac{\mathbf{v}_0 \cdot \mathbf{r}_0}{2e\xi}\left[1 - \frac{\boldsymbol{\omega}_0 \cdot \mathbf{h}_0}{\mu}r_0\right]\right\}$$

c-ii: If $\xi = \frac{1}{2}\mathbf{v}_0^2 + (\boldsymbol{\omega}_0, \mathbf{r}_0, \mathbf{v}_0) + \frac{1}{2}(\boldsymbol{\omega}_0 \times \mathbf{r}_0)^2 - \frac{\mu}{r_0} > 0$ and $\mathbf{h}_0 = \mathbf{0}$, the motion on the trajectory is described by

$$\mathbf{r}(t) = \frac{\sqrt{2\xi}}{n}[\cosh E(t) - 1]\mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0}$$
$$\mathbf{v}(t) = \frac{\sqrt{2\xi}\sinh E(t)}{\cosh E(t) - 1}\mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0} - \frac{\sqrt{2\xi}}{n}[\cosh E(t) - 1] \times \left[\boldsymbol{\omega} \times \mathbf{R}_{-\omega(t)}\frac{\mathbf{r}_0}{r_0}\right] \tag{64}$$

where $E(t)$ is defined by

$$\sinh E(t) - E(t) = n(t - t_0) + \sinh E_0 - E_0$$
$$E_0 = \sinh^{-1}\left(n\frac{\mathbf{v}_0 \cdot \mathbf{r}_0}{2\xi}\right), \qquad t_P = t_0 - \frac{1}{n}(\sinh E_0 - E_0)$$

The formulas (64) work for $t \in [t_0, t_P)$ if $\mathbf{r}_0 \cdot \mathbf{v}_0 < 0$ and for $t \in [t_0, +\infty)$ if $\mathbf{r}_0 \cdot \mathbf{v}_0 > 0$.

3. Remarks

In Sec. IV.B.2, the expressions for the position and velocity vectors depend on two essential time-dependent maps: a scalar map, analogous to the inertial eccentric anomaly $E = E(t)$, and a tensorial map, the proper orthogonal tensorial map $\mathbf{R}_{-\omega}$.

By using various expansions of map $E = E(M)$ [10,30], approximate solutions to Kepler's problem in rotating reference frames can be obtained with the time as an independent variable. Here E denotes the classic eccentric anomaly and $M = n(t - t_P)$ stands for the classic mean anomaly.

To obtain a time depending expression for $\mathbf{R}_{-\omega}$ the Darboux equation (7) must be solved. Explicit solutions to Eq. (7) are given in Sec. II.B in three particular cases. In case no solution with time as an independent variable is obtained explicitly, there are several other methods to determine the rotation tensor $\mathbf{R}_{-\omega}$ such as interpolation or spline approximation [28].

The vectorial closed form expressions for the law of motion and the velocity vector from Sec. IV.B do not depend on the choice of the coordinate system in the rotating reference frame. Once a coordinate system (Cartesian, cylindrical, spherical) is chosen, the parametric equations for the law of motion and the velocity are obtained using 3×1 matrices for vectors and 3×3 matrices for tensors.

V. Conclusions

An exact solution to Kepler's problem in rotating reference frames was presented. The main instrument used in this paper is represented by orthogonal and skew-symmetric tensorial maps. As a result, the Keplerian motion in rotating reference frames can be analyzed by decomposing it into two independent motions; one is given by the characteristics of the classic Kepler's problem and the other one by the angular velocity of the rotating reference frame. The motion can be visualized as a classic Keplerian motion in a variable plane (or on a variable straight line). The shape of the trajectories may have surprising aspects. They may be bounded or unbounded, closed or open curves, depending on both the initial conditions and the instantaneous angular velocity of the rotating reference frame. In some particular cases, the trajectories may even be planar curves. Explicit solutions for the law of motion and the velocity are obtained by using the vectorial regularization of Kepler's problem in rotating reference frames introduced in this paper. Necessary and sufficient conditions for periodicity of the motion are deduced. The results hold true for any differentiable instantaneous angular velocity.

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